GROUP THEORY UNIT: TD

THINK A DOT GROUP

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Prerequisites This unit assumes you are familiar with the Think–a–Dot device and have solved some puzzles which you can obtain from A computer program by Don Love: webster.edu/~lovedo/thinkadot The purpose of this unit is to develop a group structure which can be used to analyze questions related to Think–a–Dot puzzles. The main mathematical background you need for this unit is a good understanding of some basic ideas about groups. In particular we presuppose the concepts of a cyclic group, a group of bijective functions under composition, group generators, order, cancellation law, direct sum, morphism.

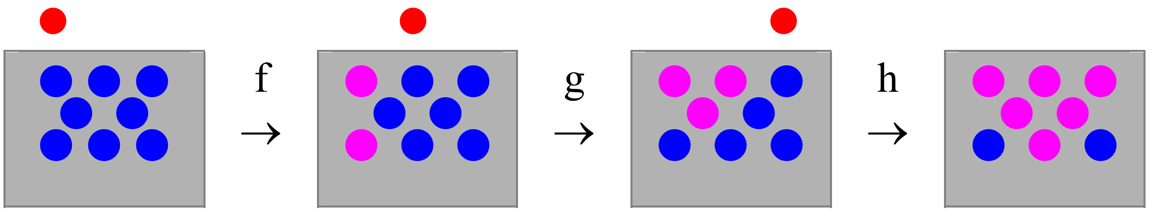
This unit contains a supplementary appendix relating the analysis of D to the fundamental decomposition theorem for abelian groups. Generator conditions for D are given without reference to the Think–a–Dot device, and we show how these condition allow D to be decomposed into $Z_8 x Z_2 x Z_8$. We then indicate how this idea can be generalized. A reader who is only interested this particular topic can be read the appendix without reference to the rest of the paper.

Reference for Main Notation

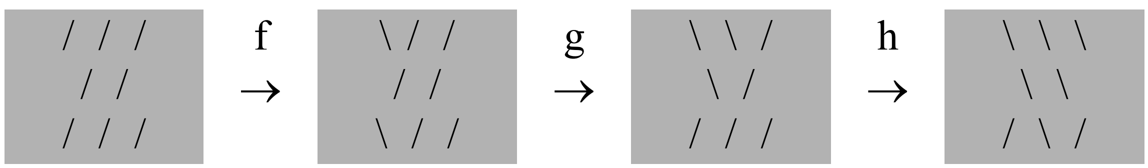
- Z_2: the additive group of the integers (mod 2)
- Z_8: the additive group of the integers (mod 8)
- ⊕: addition (mod 8) in most contexts
- +: ordinary addition in some contexts
- f: marble drop function in left hole
- g: marble drop function in middle hole
- h: marble drop function in right hole
- +: function composition in some contexts
- +: sum of 2 groups
- ⊕: direct sum of 2 groups
- D: the group generated by {f, g, h}
- P: set of think–a–dot patterns resulting from D
- ô: order of a group or group element

SECTION 0 THE PROBLEM

<u>The Think–a–Dot Device</u> Recall that this device has 3 holes at the top thru which the red marble can be dropped. It then falls thru gates inside the device, each set to send the marble left or right. A marble does not hit a gate in level 2 when dropped thru a side hole whose gate at level 1 is set toward the wall. Unless otherwise specified, it should be understood that we have started with the initial pattern in which all gates set to the left. As a marble goes thru a gate it sets the gate in the opposite direction. On the outside in front of each gate is a blue dot if the gate is set left or a pink dot if it is set right. Let f, g, h denote the marble drops thru the left hole, middle hole, right hole respectively. The diagram below shows what you would see f followed by g followed by h.



The unseen inside gate settings are as follows.



Since any setting can be in one of 2 states, it is convenient to think in terms of bits rather than colors. A zero indicates that a gate is set so a marble goes left. A one indicates that it is set so a marble goes right. Thus you can think of the result of applying these marble drops in terms of the bit patterns below.

0 0 0	f	1 0 0	g	1 1 0	h	1 1 1
0 0	→	0 0	→	1 0	→	1 1
0 0 0		1 0 0		0 0 0		0 1 0

<u>Ac00</u> Show the bit patterns which result from applying the sequence h, h, f, g, g.

<u>Main Problem</u> Let P denote the set of patterns which can be obtained by some sequence of marble drops.

1. Give a simple criteria for determining when a pattern x is in P.
2. For any x∈P, determine a sequence of marble drops that produces x.
3. Find the minimum number of drops needed to obtain x and a minimal sequence that produces x.
4. Given any other natural number j determine if x can be obtained using a sequence of exactly j marble drops. If so, give such a sequence.

<u>Re00</u>	000	h	001	h	000	f	100	g	010	g	100
	00	→	01	→	01	→	01	→	11	→	10
	000		010		011		111		011		010

Supplementary Problem Let y and x be any patterns. Is there a sequence of drops that changes y to x. If so find such a sequence. Also find the smallest number s of drops needed to obtain x from y a sequence of length s that does this. Given any other natural number j determine if x can be obtained from y using a sequence of exactly j marble drops. If so, give such a sequence.

The Group D Each of the marble drops f, g, h has a definite effect on each pattern in P, and we will also the symbols f, g, h to denote the functions from P to P that correspond to marble drops f, g, h. The structure D of functions generated by composing zero or more of these functions is associative and has an identity element. Section 1 and in Section 2 give 2 different ways of showing that each of f, g, h has order 8, and that these function commute. Thus D is shown to be an abelian group. In Section 3 we use the properties of this group to solve parts 3 and 4 of the main problem. This is also relevant to part 1 of the main problem. However in Section 1 we suggest a solution to part 1 which does not involve D.

Marble Drops Having shown that marble f and g and h all have order 8, we focus primarily on set of marbles involving less 8 drops thru any one hole. We use the group $Z_8 \oplus Z_8 \oplus Z_8$ to name such sets. We can think of this group as acting on P, altho it will be seen that different elements of this group may perform the same action on P. This suggests a morphism from $Z_8 \oplus Z_8 \oplus Z_8$ onto D.

Notation We use additive notation for D, with left to right convention for composition. Thus g+f means do g then do f, while f+g means do f then do g. We also use the standard additive abbreviations.

0: the identity function $-$: the inverse operation, nx: n applications of any X in {f, g, h}.

Ac01 In the initial pattern both level 1 and level 3 are 000, so the Z_2 sum of these 6 bits is 0. What will the Z_2 sum of these 6 bits be after 1 marble drop? After 2 marble drops? Explain and generalize to give a necessary condition for members of P. Explain why this shows that P has at most 128 members. Give an example showing that the Z_2 sum of all 8 bit need not be 0.

Overview Section 1 presents a simple strategy for making any pattern in P. It is based on the observation that we can obtain any bit pattern in level 1 using some subset of {f, g, h}, that further use of an even number of drops in a hole will not alter any bit setting in level 1, nor will 4 drops in a hole alter any bit setting in level 2. We suggest you try to discover and describe such a strategy before reading our formulation of it. This strategy yields 128 patterns, thus the necessary condition from Ac01 is also sufficient. However this strategy gives only a limited type of solution to part 2 of the main problem because the marble drops needed can only be determined as you use them.

Section 2 uses a code to solve part 2 of the main problem. Unlike the solution from Section 1, this solution allows us to calculate the marble drops needed prior to dropping any marbles. This solution does not depend on any ideas in Section 1, so you can work these sections in either order.

Section 1 and 2 both show that D is a group with certain specific properties. Section 3 uses these properties to deduce the structure of D and then relates this to the solutions to parts 3 and 4 of the main problem.

Re01 Each marble drop must change exactly 1 gate in level 1 and exactly 1 gate in level 3, so the Z_2 sum of these 6 bits remains 0 no matter how many marble drops are used. This give a necessary condition for patterns in P:

$x \in P \Rightarrow$ the bit sum in levels 1 and 3 is 0 (mod 2)

There are 8 gates, giving $2^8 = 256$ imaginable patterns. Since exactly half of these imaginable patterns have 0 for the Z_2 sum of the top and bottom levels, P has at most 128 elements. Ac00 gave an example of a pattern in P for which the Z_2 sum of all 8 gates is not 0.

SECTION 1 TOP DOWN STRATEGY FOR MAKING ELMENTS OF P

Remark This section presents a simple strategy for making any pattern in P, using the ideas involved to show that D is an abelian group. First observe that we can obtain any pattern of bits in level 1 using some subset of {f, g, h}, and that further use of an even number of drops will not alter these bits.

Ac10 Starting with any pattern, 2f changes the left level 1gate twice and the left level 2 gate once. Thus 4f changes these gates an even number of times. 4f also changes the left 3^{rd} level gate 3 times and the middle level 3 gate once. Indicate the number of times each gate is changed by 4g. Do the same for 4h.

Example Four drops thru the same hole changes gates only in level 3. This observation can be used to make the pattern with level 1 as 111, level 2 as 00, level 3 as 010. To obtain the desired first level of 111 we need to change each bit, so begin with f+g+h. After this the second level of 11 differs from the desired second level of 00 in both bits. 2g leaves level 1 alone but reverses both bits in level 2. After f+g+h+2g the third level is 001, which differs from the desired bottom level 010 in its last 2 bits. 4h changes these 2 bits but leaves levels 1 and 2 alone. Thus d = f+g+h+2g+4h will give this pattern.

000	f+g+h	111	2g	111	4h	111
00	→	11	→	00	→	00
000		010		001		010

Ac11 Apply a strategy like the above to find a function giving 101 00 011. Do the same for 101 11 011 and for 001 11 010. Before reading the strategy below, see if you can describe a this strategy.

Top Down Strategy The strategy just used for 111 00 010 can be used to find a element d for any of the 128 patterns in which the Z_2 sum of the level 1 and level 3 bits is 0. First use single drops to obtain the desired level 1. If level 2 needs to be changed select an element of {2f, 2g, 2h}. If needed, select an element of {4f, 4g, 4h} to obtain the desired level 3. In more detail, let x be a pattern you are to obtain.

(1) Create a pattern x_1 by using nf+kg+mh where n, k, m are the left, middle, right entries in the first level of x.

(2) Create a pattern x_2 from x_1 as follows: Let y be the bit by bit Z_2 sum of the second levels of x and x_1. If y = 00 do nothing. If y = 10 use 2f. If y = 11 use 2g. If y = 01 use 2h.

(3) Create the pattern x from x_2 as follows: Let z be the bit by bit Z_2 sum of the third levels of x and x_2. If z = 000 do nothing. If z = 110 use 4f. If z = 101 use 4g. If z = 011 use 4h.

Remark Step (3) can be used iff the bit sum s for z is 0, which happens iff the bit sum of all the level 1 and level 3 bits in x and x_2 is 0. Since $x_2 \in P$, we have s = 0 iff $x \in P$. Thus the necessary condition given earlier is sufficient.

Solution to Main Problem Part 1 $x \in P \Leftrightarrow$ the bit sum in levels 1 and 3 is 0 (mod 2)

Remark In finding d by the top down strategy, we first choose at most 3 drops, then at most 2 drops, then at most 4 drops; giving at most 9 marble drops. The pattern 111 00 010, is an example in which using this strategy involves for 9 drops. In Section 3 use the structure of D to show that this pattern cannot be obtained with fewer than 9 marble drops, however that there are patterns, such as 101 00 110, for which the top down strategy does not yield the minimal number of marble drops (see Ac13).

Re10 4g : 1^{st} level 0 4 0, 2^{nd} level 2, 3^{rd} level 1 2 1 4h: 1^{st} level 0 0 4, 2^{nd} level 0 2, 3^{rd} level 0 1 3

Re11

000	f+h	101	2h	101	4f	101
00	→	01	→	00	→	00
000		110		101		011

000	f+h	101	2f	101	4g	101
00	→	01	→	11	→	11
000		110		110		011

000	h	001	2f	001
00	→	01	→	11
000		010		010

Remark Given any $x \in P$ there is a function d such that d(000 00 000) = x, and the top down strategy gives a name for d. In a sense this also solves part 2 of the main problem. However this top down strategy does not allow us to predict d before dropping any marbles. The solution to part 2 given in the Section 2 depends on using a code which will allow us to easily describe the actions of f and h on P. This gives a way to calculate d prior to any marble drops.

We now turn some results about the algebra of D, the first of which should be apparent from Ac10.

Notation ô denotes the function which maps an element of D to its order.

Ac12 Explain why ôf = 8. Find ôg and ôh.

Inverses Since 8f = 0, 7f is the inverse of f. Likewise −h = 7h and −g = 7g. Thus D is a group.

Observation Each gate is changed 4 times by 4f+4g+4h; as may be seen by adding the number of gate changes indicated in Ac10. From this one might suspect that 2f+2g+2h would change each gate twice. In fact dropping 2 marbles in each hole in any order will change each gate exactly twice. Clearly this is the case for each 1st level gate. For the left gate in level 2, exactly one of the drops f will change it and exactly one of the drops g will change it, so it will also be changed twice. The left gate on level 3 will be changed once from a marble coming from the left gate on level 1 and once from a marble coming from the left gate on level 2. Similar analysis applies to the right gates on levels 2 and 3. Since 6 marble go thru gates on level 3, the middle gate must also change twice.

Claim D is an Abelian Group.

Proof By the preceding observation, f+g+f+g+2h = 0 = g+f+f+g+2h, giving f+g = g+f. Similar reasoning gives g+h = h+g and f+h = h+f, so the group D is abelian.

Remark Since D is abelian it only the number of marble drops of each type that is relevant to obtaining a pattern. For instance if x is 111 000 10 we can use f+3g+5h instead of f+g+h+2g+4h to obtain x. From an algebraic perspective this means that f+3g+5h = f+g+h+2g+4h, a result which follows easily since D is abelian. The fact that 8f = 0 means we could also use 9f+3g+5h to obtain x, and expanding on this idea there are an unlimited number of ways to obtain x.

Re12 It should be clear from Ac10 that 8f changes all gates an even number of times 8f = 0. Since 4f changes at least one gate an odd number of times, $4f \neq 0$. Thus ôf = 8. Similar analysis holds for g and h.

SECTION 2 CALCULATING THE EFFECT OF MARBLE DROPS

Notation The focus of this section is to find Z_8 formulas for calculating the effects of f and h. This will allow us to obtain half the pattern in the possible set of patterns P. Using one application of g along with these formulas will allow us to obtain the other half of the patterns in P. We also use the ideas involved to show that D is an abelian group in a way that does not depend on Section 1.

A Code For P From Ac10 we know that for any x in P the sum of the bits in the top and bottom rows is 0 (mod 2). Thus the middle bit in level 3 is the (mod 2) sum of these other 5 bits. Thus we can ignore this bit when trying to obtain a pattern. This is convenient because it is the only bit affected by all 3 types of marble drops. This allows us to focus only on the left side bits when examining the effect of f . and only on the right side bits when examining the effect of h. One compact way to describe the effects of f is to think of the left bits as binary codes for numbers in Z_8. To be specific we still code each blue dot with a zero, but we code a pink dot in the top level as a one, a pink dot in the middle level as a two, a pink dot in the bottom level as a one. We then add these to code the left side bits, giving a number in Z_8. A similar remark applies to the right side bits. For now we ignore the middle dot in the top row and focus only on patterns where this dot is blue. Since we only need to consider one middle bit, we can simply use the letter b to name this bit. Thus we can code any such pattern as a pair (a, c) from $Z_8 \oplus Z_8$.

Example Decoding the (6,5), gives (2+4, 1+4), so (6,5) codes 001 10 111. To obtain the 1 in the middle of the bottom row bottom we use 0+0+1+1+1 (mod 2).

Ac20 Decode the following triples as bit patterns: (0,0), (5,2), (2,3)

Strategy For Solving Main Problem Part 2 Using this code, we can obtain a Z_8 formula for calculating the effect of f. To do this, see what happens when you apply f to 0c. See Ac22 for a more detailed hint. Use a similar strategy to find a formula for calculating the effects of h. Next find a formula for calculating the effects of nf+mh. Use this to give a formula which would yield a marble drop for any pattern in which the middle dot is blue.

Notation To distinguish between addition for the group D and addition for Z_8, we use $\oplus$ for Z_8 addition.

Ac21 Find a Z_8 formula for f(a, c). Hint, look at f(0,c), ff(0,c), fff(0,c), etc. Also find a Z_8 for h(a, c).

Ac22 Show f+h = h+f

Formulas for f and h Below we list the formulas for f and h along with a formula which can be derived from them for multiple use of f and h.

$f(a, c) = (a\oplus 5, c)$ $h(a, c) = (a, c\oplus 3)$ $(nf+mh)(a, c) = (a\oplus 5n, c\oplus 3m)$

Using f and h Using (nf+mh)(0, 0) = (5n, 3m), we determine how to obtain any pattern. For example, to obtain 000 11 100, first code it as (6, 2) and then solve the Z_8. equations need 5n = 6 and 3m = 2. This gives n = 6 and m = 6, so we can use 6f+6h. Latter we will see that fewer marble drops could be used.

Ac23 Tell how you could make each of the patterns below using marble drops only f and h.

100 11 111 001 10 001 100 10 001 000 00 101

Re20 (0,0): 000 00 000 (5,2): 100 01 100 (2,3): 001 11 010

Re21 f(0,c) = (5,c), f(5,c) = (2,c), f(2,c) = (7,c), f(7,c) = (4,c), f(4,c) = (1,c), f(1,c) = (6,c), f(6,c) = (3,c), f(3,c) = (0,c). Observe that for each $a\in Z_8$ we have $f(a,c) = (a\oplus 5, c)$. Starting with (a, 0) and applying h, the second member of the pairs becomes 3, 6, 1, 4, 7, 2, 5, 0. Thus $h(a, c) = (a, c\oplus 3)$

Re22 $(h+f)(a, c) = f(a, c\oplus 3) = (a\oplus 5, c\oplus 3) = h(a\oplus 5, c) = (f+h)(a, c)$, so f+h = h+f.

Re23 3f+2h, 2f+7h, 7f+5h, 4f+4h.

Ac24 Use (nf+mh)(0,0) = (5n, 3m) to show that ((5a)f+(3c)h)(000) = (a, c).

The Remaining Patterns In order to examine the patterns for which the middle dot in the top row is pink we extend the code to a triple (a, b, c) where a and c are as before and b is 0 or 1 depending on whether this dot is blue or pink. Since g(0,0,0) = (6,1,0) we can obtain any pattern in which the top middle dot is pink by using g and then using only combinations of f and h.

Ac25 Write a formula for (g+nf+mh)(0,0,0)

Remark Ac24 shows how to obtain any element in P of the form a0c. Use the result from Ac25 to give a formula for obtaining a1c from 000. We need $5n \oplus 6 = a$ & $3m = c$. Thus $n = 5a \oplus 2$ & $m = 3c$.

$$\text{This gives } (g+(5a\oplus 2)f+(3c)h+g)(0,0,0) = (a,1,c)$$

Solution to Main Problem Parts 1 and 2. These activities give the formulas below, which show at least one way to obtain any of the 128 patterns of the form (a, b, c), so by the earlier result in Section 0 the 128 pattern for which the mod 2 sum of the bits in levels 1 and 3 are exactly the patterns in P. This solve parts 1 and 2 of the main problem.

$$((5a)f+(3c)h)(0,0,0) = (a,0,c). \qquad ((5a\oplus 2)f+(3c)h+g)(0,0,0) = (a,1,c)$$

Remark The above formula does not use g to obtain patterns of the form (a,0,c). For example, it uses 6f+6h to produce (6,0,2). Since 2g(000) = (602), 6f+6h is clearly not the smallest number of drops needed to make (6,0,2). Likewise this formula use 6f+6h+g to produce (4,1,2), which could be produced using 3g. The next section explores this further.

Formula for g If the top middle is 0 and the left middle is 0 the g changes the left middle to 2 and adds 4 to the bottom left, thus $g(a,0,c) = (a\oplus 6, 1, c)$. If the top middle is 0 and the left middle is 2 the g changes the left middle to 0 and leaves the bottom left the same, thus $g(a,0,c) = (a\oplus 6, 1, c)$. Similar reasoning yield formulas involving.

$$g(a,0,c) = (a\oplus 6, 1, c) \qquad g(a,1,c) = (a, 0, c\oplus 2) \qquad 2g(a, b, c) = (a\oplus 6, b, c\oplus 2).$$

Remark Since $(6f+6h)(a, b, c) = (a\oplus 6, b, c\oplus 2)$, we have 2g = 6f+6h. Given this result, it is easy to see why 3g give the same result as g+6f+6h.

Results About D The rest of this section uses the formula for marble drops to derives the same results about D as in we derived in Section 1. From the formulas for f and h it is easy to see that $\hat{o}f = 8$ & $\hat{o}h = 8$. The formula for 2g shows that $\hat{o}g = 8$. Since 8f = 0, 7f is the inverse of f. Likewise h and g have 7h and 7g as inverses so D is a group. We can also show that D is abelian. We have already shown f+h = h+f. The cases below show f+g = g+f, and the proof of h+g = g+h is similar.

$$(f+g)(a,0,c) = g(a\oplus 5, 0, c) = (a\oplus 3, 1, c) = f(a\oplus 6, 1, c) = (g+f)(a,0,c)$$

$$(f+g)(a,1,c)=g(a\oplus 5, 1, c) = (a\oplus 5, 0, c\oplus 2) = f(a, 0, c) = (g+f)(a,1,c)$$

Ac26 Using 2g = 6f+6h, show that 2f+2g+2h = 0.

Re24 nf+mh(0, 0) = (a, c) $\Leftrightarrow$ (5n, 3m) = (a, c) $\Leftrightarrow$ a = 5n & c = 3m $\Leftrightarrow$ n = 5a & m = 3c

Re25 (g+nf+mh+g)(000) = g(6, 0, 0) = $(5n\oplus 6, 1, 3m)$

Re26 Since 2g = 6f+6h, 2g+2f+2h = 6f+6h+2f+2h. Now use the fact that D is abelian and that 8f = 0 and 8h = 0.

SECTION 3 THE MINIMUM NUMBER OF DROPS FOR A PATTERN

Normal Names for Elements of D For n, k, m∈Z, we let the triple [n, k, m] denote the sequence of n drops thru the left hole, k drops thru the middle hole, m drops thru the right hole. This triple will also be used to denote the element nf+kg+mh of D. While different triples always denote different sequences of marble drops, each element of D can be named by many different triples. When n, m, k$\in Z_8$ we call the triple [n, m, k] a normal name. Since ôf = ôg = ôh = 8, a minimal sequence of marble drops for obtaining a pattern will have a normal name.

Ac30 The top down strategy gives the 8 marble drops [1, 0, 7] to produce 10 100 110. Use 6h = 2f+2g to find a way to obtain x involving only 6 drops. Find 2 other normal names for d.. Find 4 normal names for the element that produces 101 11 011, for 001 11 010.

Alternate Normal Names The idea from Ac30 allows us to find 4 normal names for any element of D. We now develop further results about the structure of D which allow us to prove that each element of D has exactly 4 normal name, and hence solve part 3 of the main problem.

Notation Σd denotes the subgroup generated by d.

Remark This section depends on many of the results from Sections 1 and 2. In particular it depends on the following information about D.

- D is an abelian group with D = Σf+Σg+Σh, but this sum is not direct.
- Each of f, g, h has order 8, and furthermore 2f+2g+2h = 0.

Ac31 Show that Σf+Σh is direct and thus has exactly 64 elements.

A Generator of Order 2 Let e = f+g+h. By the preceding observation 2e = 0. Furthermore g = 7f+e+7h, so D = Σf+Σe+Σh.

Structure Claim D = Σf⊕Σe⊕Σh, so ôD = 128.

Proof By Ac31, Σf∩Σh = {0}. Since Σe = {0, e} and e changes bit b, but neither f nor h change bit b, Σe∩Σf = {0} and Σe∩Σh = {0}.

Note By what we have shown, D is isomorphic to $Z_8 \oplus Z_2 \oplus Z_8$.

Claim Let β denote the map from D to P defined by βd = d(000). β is a bijection .

Proof By definition of P and β, β maps D onto P. To show that β is 1 to 1, assume $\beta d_1 = \beta d_2$ for some d_1, $d_2 \in D$. This gives $d_1(000) = d_2(000)$. Application of $-d_2$ gives $(-d_2+d_1)(000) = 000$. Since 0 is the only element of D that maps 000 to 000, $-d_2+d_1 = 0$, and hence $d_1 = d_2$. Thus β is a bijection.

Claim Each element of D has exactly 4 normal names.

Proof Since each triple from $Z_8 \oplus Z_8 \oplus Z_8$ is a normal name, altogether there are 512 normal names. Using 2f+2h+2g = 0, we have shown in both Sections 1 and 2 that each element of D has at least 4 normal names. Since ôD = 128 this gives exactly 4 normal names for each element of D. Re30 d = f+7h = f+h+6h = f+h+2f+2g = 3f+2g+h [3, 2, 1] 6 drops

d = 3f+2g+h+(2f+2g+2h) = 5f+4g+3h [5, 4, 3] 12 drops
d = 5f+4g+3h+(2f+2g+2h) = 7f+6g+5h [7, 6, 5] 18 drops

For 10111011: [3, 4, 1] [5, 6, 3] [7, 0, 5] [1, 2, 7]. For 00111010: [2, 0, 1] [4, 2, 3] [6, 4, 5] [0, 6, 7]

Re31 None of the elements of Σh change any of the left gates. 0 is the only such element of Σf and hence the only element of Σf that belongs to Σh. A similar argument shows that 0 is the only element of Σh that belongs to Σf. Since Σf and Σh each have 8 elements there are 64 elements Σf+Σh, namely any element of the form nf+mh with n, m$\in Z_8$.

Name Sizes Let σ{n, k, m} = n+k+m in Z. We call this the names size of [n, k, m]. A minimal name for an element of D is one with the minimal size. Clearly only a normal name can be a minimal one.

Example Let d = [7, 3, 4]. Since 2e = 0, d = d+2e = 9f+5g+6h. Since 8f = 0, d = [1, 5, 6]. Adding 2e again gives d = [3, 7, 0], and once more gives d = [5, 1, 2]. The minimal name of this element is [5, 1, 2], and listing its normal names in order of size we have: [5, 1, 2], [3, 7, 0], [1, 5, 6], [7, 3, 4]; with sizes 8, 10, 12, 14. Since it is always possible to drop 8 more marbles without changing a pattern the possible names sizes for d are all even numbers greater than 6.

Example If d = [7, 1, 1] its normal names are [1, 3, 3], [7, 1, 1], [3, 5, 5], [5, 7, 7]; with sizes 7, 9, 13, 19. Since it is always possible to drop 8 more marbles without changing there are names of size 15 and 17. Continued use of 8 drops with those of sizes 13, 15, 17, 19 gives names sizes 21, 23, 25, 27. Thus d has name sizes of 7 and 9 and all odd numbers greater than 11.

Ac32 List the normal names and their sizes in order of size for each of the elements of D given below. Also give all possible name sizes. (1) 3f+5g+1h (2) 2f+4g+5h (3) 3f+6g+2h (4) 5f+4g+6h

Reduced Names Let min[n, k, m], mid[n, k, m], max[n, k, m] be the values of n, m, k in order of size. For example, min[5, 1, 3] = 1, mid[5, 1, 3] = 3, max[5, 1, 3] = 5. We call a name [n, k, m] reduced if is satisfies condition below.

min[n, k, m] < 2 & mid[n, k, m] < 4 & max[n, k, m] < 6

Claim [n, k, m] is a minimal name ⇔ [n, k, m] is reduced

Proof We first use the top down strategy to show that each element d of D has a reduced name. This strategy gives a reduced name except when the 2nd and 3rd stages involve 6 drops thru the same hole H. In this case the other holes were each used at most once. Replacing the 6 drops thru H with 2 drops thru each of the other 2 holes will give a name with a 1or 0 for hole H and 3 or less for the other holes. We next show that a reduced name [n, m, k] is minimal. Consider the five different ways a [n, m, k] may be a reduced name, observing that in each case the other normal names have larger sizes. Let s = n+k+m. While not part of the proof we also list all possible name sizes in relation to the minimal size s.

min mid max	Normal Name Sizes	Possible Name Sizes
{0, 1} {2, 3} {4, 5}	s, s+2, s+4, s+6	s+2j
{0, 1} {2, 3} {2, 3}	s, s+2, s+6, s+12	s, s+2, s+6+2j
{0, 1} {0, 1} {4, 5}	s, s+4, s+6, s+10	s, s+4+2j
{0, 1} {0, 1} {2, 3}	s, s+6, s+10, s+12	s, s+2, s+6+2j
{0, 1} {0, 1} {0, 1}	s, s+6, s+12, s+18	s, s+6, s+8, s+12+2j

Solution to Parts 3 and 4 of The Main Problem One way to solve part 3 of the main problem is to use the variation of the top down strategy, as indicated in the proof above. This solution does not allow us to specify the number of drops needed without first making some marble drops. To solve part 3 without trying any marble drops, recall from Section 2 the 4 normal names for d if d(000) = (a, b, c). The last of these is clearly not reduced, so check the others until you find one which is reduced.

(2b⊕5a, b, 3c) = (2b⊕5a⊕2, b⊕2, 3c⊕2) = (2b⊕5a⊕4, b⊕4, 3c⊕4) = (2b⊕5a⊕4, b⊕6, 3c⊕6)

To solve part 4 just note the list of possible name sizes in the response to Ac32.

Re32	Normal Names	Their Sizes	Possible Name Sizes
	[3, 5, 1], [1, 3, 7], [7, 1, 5], [5, 7, 3]	9, 11, 13, 15	9+2j
	[0, 2, 3], [6, 0, 1], [2, 4, 5], [4, 6, 7]	5, 7, 11, 17	5, 7, 11+2j
	[1, 4, 0], [5, 0, 4], [3, 6, 2], [7, 2, 6]	5, 9, 11, 15	5, 9+2j
	[1, 0, 2], [3, 2, 4], [7, 6, 0], [7, 6, 0]	3, 9, 13, 15	3, 6 9+2j

EXERCISES AND PROBLEMS FOR ALL THREE SECTIONS

Ex1 Represent the following as ordered triples from $Z_8 \oplus Z_2 \oplus Z_8$. Give minimal way to obtain each.

101 01 011 100 10 100 001 00 010 100 11 001 010 11 100 110 10 010 111 00 111

Ex2 The following ordered triples 305, 502, 203, 713, 416, 311, 605, 517 represent which patterns. Give the minimal way to obtain each of these.

Ex3 (g+nf)000 = (a,1,0) where $a = 5n \oplus 6$. Show in detail how to solve this equation to obtain $n = 5a \oplus 2$.

Ex4 Represent each element below the form nf+ke+mg, where $n, m \in Z_8$ and $k \in Z_2$.

3f+6g+2h 5f+4g+6h 3f+5g+h f+g+h 2f+4g+6h 0

Prove that each element of D has a unique name of that form. Prove that α(nf+ke+mg) = [n, k, m] gives an isomorphism α from D onto $Z_8 \oplus Z_2 \oplus Z_8$.

Ex5 Let nf+kg+mh be a normal name of some d in D. This name is called maximal if it is the normal name of d with the largest size. Let p, q, r be the values of n, m, k with $p \le q \le r$. Show that for the maximal name $p \ge 2$, $q \ge 4$, $r \ge 6$.

Ex6 Let $abc \in P$ and d(000) = abc. What is −d(abc)? If d = [n, k, m], what is a normal name for −d. Either prove or give a counter example to the following claim.

[n, k, m] is the maximal normal name for d $\Leftrightarrow$ (−n, −m, −k) is the minimal name for −d

Ex7 Let abc, $pqr \in P$. Find d such that d(abc) = pqr.

Supplementary Problem 2 Let $y \notin P$. Investigate which patterns can be obtained by application of marble drops D to y, and how they can be obtained.

An1 107, 700, 001, 302, 212, 314, 416, 111 505, 300, 003, 706, 416, 114, 612, 713

An2

101	100	001	111	010	111	001	111
10	01	11	11	01	10	10	01
011	100	010	100	111	010	111	111

707 106 201 511 512 112 607 315

An3
$5n \oplus 6 = a$
$5n \oplus 6 \oplus 2 = a \oplus 2$
$5n = a \oplus 2$
$5(5n) = 5(a \oplus 2)$
$(5 \bullet 5)n = 5a \oplus 5 \bullet 2$
$n = 5a \oplus 2$

APPENDIX: DECOMPOSITION THEOREM FOR ABELIAN P–GROUPS

Notation Let D be any additive abelian group. ΣB denotes the subgroup generated by a set B of elements in D. ôB is the order of ΣB. Σb is an abbreviation for Σ{b}, and ôb is the order of Σb. The symbol ≅ is brief for 'is isomorphic to'.

Perspective The main text uses an abelian group D to analyze some questions about the Think–a–Dot devise. Details of this application are irrelevant to this appendix. What is relevant is the use of $Z_8 \oplus Z_8 \oplus Z_8$ to name elements of D and the conditions on D indicated below.

(1) ôD = 128   (2) D = Σ{f, g, h}   (3) ôf = ôg = ôh = 8   (4) 2f+2g+2h = 0

These conditions can be used to show that each element of D can be represented in the form mf+ng+ie, where e = f+g+h and m, n∈Z_8 & i∈Z_2. Since there are exactly 128 such representations, each element of D has a unique representation of this form. Thus D = Σf⊕Σg⊕Σe, i.e.

$$D \cong Z_8 \oplus Z_8 \oplus Z_2.$$

My observation that $Z_8 \oplus Z_8 \oplus Z_8$ was the natural way to name marble drops but that each element of D had exactly 4 such names suggested the morphism α below. Since ô($Z_8 \oplus Z_8 \oplus Z_8$) > ôD, α is not an isomorphism. In fact by (2) and (4), kernel(α) = {[0, 0, 0], [2, 2, 2], [4, 4, 4], [6, 6, 6]}.

The map α : $Z_8 \oplus Z_8 \oplus Z_8 \to D$, where α[m, n, j] = mf+ng+jh, is a morphism onto D. (see Exercise 0b).

Since D is isomorphic to $Z_8 \oplus Z_8 \oplus Z_2$ the existence of a morphism from $Z_8 \oplus Z_8 \oplus Z_8 \to D$ is apparent without this observation. However it was this specific morphism that provided me a new perspective on the fundamental decomposition of abelian groups. Before turning to the proof this suggested, we will look at some examples. The first example illustrates that conditions like (1), (2), (3) imply some additional condition on generators, such as (4). However this example might suggest that they uniquely determine such a condition. Further examples are give a fuller perspective.

Suggestion Each example gives some conditions on an abelian group D and uses the generator conditions to indicate representations for elements of D. These representations suggest a morphism α from a direct sum of cyclic groups onto D. We denote the kernel of α as K. The example then show how to use K to find a set of generators with lower combined order than the ones given in the conditions. Read the conditions and try to work out some the details before reading the rest of the example.

Example 1 Conditions: (1) ôD = 16   (2) D = Σ{f, g}   (3) ôf=8 & ôg=4

Sketch We now show that these conditions imply (4) 4f+2g = 0 and letting e = 2f+g that D = Σf⊕Σe. This can be sketched as follows. Elements of D are of the form mf+ng with m∈Z_8 and n∈Z_4. Use redundancy to show 0 has a nontrivial representation. Use a morphism from $Z_8 \oplus Z_4 \to D$ to find it.

Details By (1) and (2) every element of D has a representation of the form mf+ng with m∈Z_8 and n∈Z_4. Since there are 32 representations of this form, some element has more than one such representation, and this means 0 has non-trivial a representation (one other than 0f +0g). Thus we must have some condition like (4) from the think-a-dot situation which allowed us to find a generator of smaller order.

The map α : $Z_8 \oplus Z_4 \to D$, where α[m, n] = mf+ng is a morphism. By (3) it is onto D. K is not trivial so K has an element of order 2. The only elements of order 2 are [4,0], [0,2], [4,2]. If [4,0] was in the kernel then 4f = 0, contradicting (3). Likewise [0,2]∉K. Thus [4,2]∈K giving 4f+2g = 0. Letting e = 2f+g, we have g = e+6f. Thus D = Σ{f, e}. Since there are exactly 16 representation mf+ne with m∈Z_8 and n∈Z_2,

$$D = \Sigma f \oplus \Sigma e \cong Z_8 \oplus Z_2.$$

Further examples shows that with like (1), (2), (3) always imply a the existence of a condition like (4). Exercise 1 shows that with only 2 generators such conditions uniquely D. To illustrate a broader perspective the rest of the examples all involve more than 2 generators.

<u>Example 2</u> Conditions: (1) ôD = 128 (2) D = Σ{f, g , h} (3) ôf = 8 & ôg = 8 & ôh = 4

By (2) and (3) each element of D has a representation of the form mf+ng+kh with $m\in Z_8$, $n\in Z_8$, $k\in Z_4$. As before 0 must have a non-trivial representation of the form mf+ng+kh.

Let α be the morphism from $Z_8\oplus Z_8\oplus Z_4$ onto D, with α[m, n, j] = mf+ng+jh and kernel K.

By (1), ôK = 2. Thus K contains [0,0,0] and exactly one of {[4,4,2], [4,4,0], [4,0,2], [0,4,2]}.

Case 1. [4,4,2]∈K. Letting e = 2f+2g+h, gives ôe = 2, h = e+6f+6g, D = Σ{f, g, d}. There are exactly 128 elements of the form mf+ng+ke, with $m\in Z_8$, $n\in Z_8$, $k\in Z_2$. Thus $D \cong Z_8\oplus Z_8\oplus Z_2$.

Case 2. [4,4,0]∈K. With e = f+g: g = e+7f, D = Σ{f, h, e}. Since [2,2,0]∉K, ôe = 4, $D \cong Z_8\oplus Z_4\oplus Z_4$.

Cases 3 and 4. See Exercise 2.

<u>Comment</u> In Examples 1 and 2 the generator conditions suggested twice as many names as elements. In any such example the order of the kernel of the map from the naming group to D is 2^1, and we call say that the name generation excess is 1. In such cases the non-trivial element of K can be use to find a generator set for D with a no excess of names. In the next example the order of the kernel of the map from the naming group to D is 2^2, and we call say that the name generation excess is 2.

<u>Example 3</u> Conditions: (1) ôD = 128 (2) D = Σ{f, g , h} (3) ôf = 8 & ôg = 8 & ôh = 8

Let α be the morphism from $Z_8\oplus Z_8\oplus Z_8$ onto D, with α[m, n, j] = mf+ng+jh and kernel K.

By (1), K has a element of order 2. Thus K contains at least one of {[4,4,4], [4,4,0], [4,0,4], [0,4,4]}.

Case 1. [4,4,4]∈K. Letting e = f+g+h, we have h = e+7f+7g, D = Σ{f, g, e}, 4e = 0. Since D ≠ Σ{f, g}, e ≠ 0. Thus ôe = 2 or ôe = 4. ôe = 2 ⇒ $D \cong Z_8\oplus Z_8\oplus Z_2$. If ôe = 4 we have the situation of Example 2.

Case 2. [4,4,0]∈K. Let e = f+g, giving g = e+7f, D = Σ{f, h, e}. Since 4e = 0 & e≠0, ôe = 2 or ôe = 4. If ôe = 2 then $D \cong Z_8\oplus Z_8\oplus Z_2$. If ôe = 4 we have the situation of Example 2.

Cases 3 and 4 follow by symmetry.

<u>Comment</u> In all the above cases the name excess of 2 suggests a morphism whose kernel has order 4. Using an element of order 2 from K, we find a generating set with smaller name generation excess. However you might observe that two possibilities occur. We may find a generator set with 0 excess giving a direct product or we may only reduce the excess from 2 to 1. In the latter case we had to refer to the preceding example to complete the decomposition. While this is all that is relevant to the proof of the decomposition theorem, we have included Exercise 3 to supply some additional perspective. The name generation excess is 3 in Example below. We merely show how to find a generator set with smaller excess. It might take 2 more applications of the process to find a decomposition.

<u>Example 4</u> Conditions: (1) ôD = 128 (2) D = Σ{f, g , h} (3) ôf = 16 & ôg = 16 & ôh = 4

Let α be the morphism from $Z_{16}\oplus Z_{16}\oplus Z_4$ onto D, with α[m, n, j] = mf+ng+jh and kernel K.

Case 1. [8,8,2]∈K. For e = 4f+4g+h: h = e+4f+4g, D = Σ{f, g, e}, 2e = 0. If e = 0 then D = Σ{f, g}, and the name generation excess is reduced to 1, with a further reduction giving $D \cong Z_{16}\oplus Z_8$. If ôe = 2 the excess is only reduced to 2. Further reductions give either $D \cong Z_{16}\oplus Z_8$ or $D \cong Z_{16}\oplus Z_4\oplus Z_2$.

For more details on case 1, as well as an examination of the other cases see Exercise 4.

Remark The lemma and theorem below show that the strategy used in these examples applies to any abelian group whose order is a power of 2. It takes only a slight modification to prove the same results for any finite abelian p−group. Given these results it is easy to extend the result to any finite abelian groups. Merely show in the standard fashion that any finite abelian group decomposes into a direct product of abelian p−groups.

Notation D denotes an abelian group with ôD a power of 2. For a set B of generators of D, NGE(B) denotes name generating excess. That is NGE(B) is the power of 2 obtained by dividing the product of the orders of the elements in B by the order of D.

Lemma If $\Sigma B = D$ and NGE(B) > 0 then there is a C with $\Sigma C = D$ and NGE(C) < NGE(B).

Prf(when B has 3 members) Denote the elements of B as f, g, h, where 2a, 2b, 2c are the orders of f, g, h and notation is chosen so $a \geq b \geq c$. Note a, b, $c \in \{1, 2, 4, 8, ...\}$. Let $H = Z_{2a} \oplus Z_{2b} . \oplus Z_{2c}$,

Let α be the morphism from H onto D: [m, n, k] → mf+ng+kh., with kernel K

Since ôH > ôD, there is an $x \in K$ with ôx = 2. Thus $x \in \{[a, b, c], [a, b, 0], [a, 0, c], [0, b, c]\}$.

In all cases but the second there must be an equation of the form c(mf+ng+h) = 0. Let e = mf+ng+h.

h = −mf+−ng+e $D = \Sigma\{f, g, e\}$ ôe ≤ c < 2c = ôh   NGE{f, g, e} < NGE{f, g, h}

In the second case we have b(mf+g) = 0, and we let e = mf+g.

Prf(when B has k elements) Other than for notation the proof is essentially the same. Denote the elements of B as $f_1, ..., f_k$, where $2a_1, ..., 2a_k$ are their orders.

Let $H = H_1 \oplus ... \oplus H_k$, where H_i is the group of integers mod $2a_i$.

Let α be the morphism from H onto D: $[j_1, ..., j_k] \rightarrow j_1 f_1 + ... + j_k f_k$., with kernel K

Since ôH > ôD, there is an $x \in K$ with ôx = 2. x must be a tuple $[t_1, ..., t_k]$ where each t_i is either 0 or a_i . and where at least 2 of the entries are not 0.

Without loss of generality, suppose t_1 is a_1 and there is no smaller a_i in x. This gives an equation of the form: $a_1(f_i+m_2f_2+\ldots+m_kf_k) = 0$. Let $e = f_i+m_2f_2+\ldots+m_kf_k$, C = {e, f_2, ..., f_k}.

$f_1 = e +-m_2f_2+\ldots+-m_kf_k$

$D = \Sigma C$, since $B \subseteq \Sigma C$

ô(e) ≤ a_1 < $2a_1$ ≤ ô (f_1)

NGE(C) < NGE(B)

Theorem D is a direct sum of cyclic groups.

Prf Let $D = \Sigma B$ with NGE(B) as small as possible. By the preceding lemma, NGE(B) = ôD. Use the same H and α as in the proof of the lemma. This gives an isomorphism from H onto D.

Exercises and Problems

Exercise 0a Show that the conditions below imply that each element of D can be uniquely represented in the form mf+ng+ie, where e = f+g+h and m, n$\in Z_8$ & i$\in Z_2$.

(1) $\hat{o}D = 128$ (2) $D = \Sigma\{f, g, h\}$ (3) $\hat{o}f = \hat{o}g = \hat{o}h = 8$ (4) $2f+2g+2h = 0$

Exercise 0b Show that the map $\alpha : Z_8 \oplus Z_8 \oplus Z_8 \to D$, where $\alpha[m, n, j] = mf+ng+jh$, is a morphism onto D.

Exercise 1 Show the first set of conditions below imply $\hat{o}(f+g) = 4$ and thus $D \cong Z_8 \oplus Z_4$. Also show that the second set implies $D \cong Z_8 \oplus Z_2$ and that the third implies $D \cong Z_{16}$. Make some general observations about such groups when there are only 2 generators with orders that are powers of 2. Consider some examples of groups with 3 generators whose orders are powers of 3.

First set of Conditions:	(1a) $\hat{o}D = 32$	(2a) $D = \Sigma\{f, g\}$	(3a) $\hat{o}f = 8$ & $\hat{o}g = 8$
Second set of Conditions:	(1b) $\hat{o}D = 16$	(2b) $D = \Sigma\{f, g\}$	(3b) $\hat{o}f = 8$ & $\hat{o}g = 8$
Third set of Conditions:	(1c) $\hat{o}D = 16$	(2c) $D = \Sigma\{f, g\}$	(3c) $\hat{o}f = 16$ & $\hat{o}g = 8$

Exercise 2 Do cases 3 and 4 of Example 2.

Exercise 3 In Example 3 we know $\hat{o}K = 4$, but we only used the fact that K had an element of order 2. Show that either K = {[0,0,0], [0,4,4], [4,0,4], [4,4,0]} or K has an element of order 4. Without reference to Example 3, show that first case implies $D \cong Z_8 \oplus Z_4 \oplus Z_4$ and the second case implies $D \cong Z_8 \oplus Z_8 \oplus Z_2$.

Exercise 3a In Example 3 suppose K = {[0,0,0], [0,4,4] [4,0,4], [4,4,0]}, and hence $D \cong Z_8 \oplus Z_4 \oplus Z_4$. Find elements f, g, h of order 8 that generate $Z_8 \oplus Z_4 \oplus Z_4$ and that satisfy 4g+4h = 0, 4f+4h = 0, 4f+4g = 0.

Exercise 3b In Exercise 3 with K = {[0,0,0], [0,4,4] [4,0,4], [4,4,0]} we chose {h, g+h, f+h} as an independent generating set for D. Show that $D = \Sigma h \oplus \Sigma(3f+3g) \oplus \Sigma(5f+h)$. Find some other choices of independent generating sets the form {h, a, b}.

Exercise 3c In Example 3 suppose [4,6,2]$\in$K. Since K has an element of order 4, $D \cong Z_8 \oplus Z_8 \oplus Z_2$. Find elements f, g, h of order 8 that generate $Z_8 \oplus Z_8 \oplus Z_2$ and that satisfy 4f+6g+2h = 0.

Exercise 3d In Exercise 3c with K = {[0,0,0], [4,6,2] [0,4,4], [4,2,6]}. We chose {f, g, 2f+3g+h} as an independent generating set. Show that $D = \Sigma f \oplus \Sigma g \oplus \Sigma(2f+5g+7h)$. Give another such example.

Answer 0a By (3) $h = 7f+7g+e$. Thus by (2) $D = \Sigma\{f, g, e\}$. By (1) $D \neq \Sigma\{f,g\}$, so $h \neq 7f+7g$. Thus $e \neq 0$. Thus by (4) $\hat{o}e = 2$. By this and (3), each element of D can be represented in the form $mf+ng+ie$, where $m, n \in Z_8$ & $i \in Z_2$. There are 128 such representations, so uniqueness follows by (1).

Answer 0b $\alpha([m, n, j]+[a, b, c]) = \alpha[m\oplus a, n\oplus b, j\oplus c]$ where $\oplus$ is addition mod 8

$= (m\oplus a)f, (n\oplus b)g, (j\oplus c)h$ def of α

$= (mf+ng+jh)+(af+bg+ch)$ by condition (3) and commutivity

$= \alpha[m, n, j]+\alpha[a, b, c]$ def of α

Answer 1 By (1a) and (2a) elements of D can be represented in the form $mf+ng$ with $m\in Z_8$ and $n\in Z_8$. The map $\alpha : Z_8\oplus Z_8 \to D$, where $\alpha[m, n] = mf+ng$ is a morphism onto D. By (1a) K is not trivial and thus has an element of order 2. By (3a) neither [4,0] nor [0,4] is in K. Thus $[4, 4]\in K$, giving $4f+4g = 0$. Since $[2,2]\notin K$, $\hat{o}(f+g) = 4$. Letting $e = f+g$, $g = e+7f$. Thus $D = \Sigma\{f, e\}$. Since there are exactly 32 representation of the form $mf+ne$ with $m\in Z_8$ and $n\in Z_4$, $D = \Sigma f\oplus\Sigma e \cong Z_8\oplus Z_4$.

By (1b) and (2b) elements of D can be represented in the form $mf+ng$ with $m\in Z_8$ and $n\in Z_8$. The map $\alpha : Z_8\oplus Z_8 \to D$, where $\alpha[m, n] = mf+ng$ is a morphism onto D. By (1b), $\hat{o}K = 4$ and thus has an element of order 2. By (3a) neither [4,0] nor [0,4] is in K. Thus [4, 4] is the only element of K having order 2. This implies $K = \Sigma[2,2]$ or $K = \Sigma[2,6]$. Thus either $\hat{o}(f+g) = 2$ or $\hat{o}(f+3g) = 2$. In the first case let $e = f+g$. In the other case let $e = f+3g$. In either case $D = \Sigma\{f, e\}$. Since there are exactly 16 representation of the form $mf+ne$ with $m\in Z_8$ and $n\in Z_4$, $D = \Sigma f\oplus\Sigma e \cong Z_8\oplus Z_2$.

The third set gives $\hat{o}K = 8$, with [8, 4] the only element of order 2 in K. This implies $K = \Sigma[2,m]$ for some $m\in\{1, 3, 5, 7\}$. In any of these cases, $g\in\Sigma f$, and hence $D = \Sigma f \cong Z_{16}$.

In general consider conditions where $m \geq n \geq k \geq 2$ and these numbers are powers of 2.

Conditions: (1a) $\hat{o}D = m$ (2a) $D = \Sigma\{f, g\}$ (3a) $\hat{o}f = n$ & $\hat{o}g = k$

If $m = n$ then $D \cong Z_n$. If $m>n$ $D \cong Z_n \oplus Z_j$. where $j = m/n$. Likewise for powers of 3 or any other prime.

Answer 2

Case 3. $[4,0,2] = 2[2,0,1]\in$kernel(β). For $e = 2f+h$; $\hat{o}e = 2$, $h = e+6f$, $D = \Sigma\{f, g, e\}$. Thus $D \cong Z_8\oplus Z_8\oplus Z_2$.

Case 4. $[0,4,2] = 2[0,2,1]\in$kernel(β). For $e = 2g+h$; $\hat{o}e = 2$, $h = e+6g$, $D = \Sigma\{f, g, e\}$. So $D \cong Z_8\oplus Z_8\oplus Z_2$.

Answer 3 Suppose K = {[0,0,0], [0,4,4] [4,0,4], [4,4,0]}. Choosing e=g+h & d=f+h gives $D = \Sigma\{h, d, e\}$. Since $4e = 0$ and $2e\neq 0$ and $4d = 0$ and $4e\neq 0$, we have and $\hat{o}d = 4$ and $\hat{o}e = 4$. Thus $D \cong Z_8\oplus Z_4\oplus Z_4$.

Now suppose K has an element x of order 4, for instance [4, 6, 2]. Choosing $e = 2f+3g+h$ we have $2e = 4f+6g+2h = 0$. Since $\hat{o}e = 2$, $D \cong Z_8\oplus Z_8\oplus Z_2$. This can be generalized for any x of order 4. x has no odd entries, and at least one of them is 2 or -2. Thus either x or -x is of the form [2a, 2b, 2c] where at least one of a, b, c is 1. Suppose c = 1. Let $e = af+bg+h$. $\hat{o}e = 2$ and $D = \Sigma\{f, g, e\}$. Similar results follow if a = 1 or b = 1.

Answer 3a Many possibilities, one being f = [1,0,0], g = [0,1,0], h = [6,5,1].

Answer 3c Many possibilities, one being f = [1,0,0], g = [7,0,1], h = [1,1,7].